ON THE GEOMETRY WHOSE ELEMENT IS THE 3-POINT OF A PLANE*

BY

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The study of the 3-points of a plane is a part of the study of the cubic curves of the plane, but it is a part of special interest.

The sections 1–3 of this investigation, dealing with fully and triply perspective triangles, are mainly recapitulatory. The mapping of § 3 is discussed synthetically by KANTOR.† But it is so much more easily grasped by means of equations that I have not scrupled to repeat a part of KANTOR's argument, with dualistic apparatus. Passing to unrestricted 3-points, the mapping is not carried out, for it seems necessary first to work out (§§ 4, 5) a cubic curve arising from two 3-points, and this curve leads (§§6, 7) to a phenomenon which seems fundamental, namely, that the 3-points of a plane fall in general into sets of three.

§ 1. *Fully perspective triangles.*

Three points of a plane are called a 3-point. The case of points on a line is degenerate and is excluded unless specially mentioned.

We denote a 3-point by a Roman capital, and there will be no ambiguity in denoting the triangle of the 3-point, and the 3-line of the 3-point, by the same capital, though a stricter notation would be that of a Greek capital for the 3-line.

We note first the fully perspective triangles, that is triangles sixway perspective. These are well known as arising from the flex-configuration of a cubic curve. They occur in *tetrads*, or in sets of four.

The polar points of any line as to the 4 are on a line and form a self-apolar set.

§ 2. *Triply perspective triangles.*

It is well known that if $a_1 a_2 a_3$ is perspective with $b_1 b_2 b_3$ and with $b_2 b_3 b_1$ then it is perspective with $b_3 b_1 b_2$. A convenient proof is to identify the theorem with the theorem that a 3-point has a circumcenter.

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† Crelle's Journal, vol. 95 (1883), p. 147.

For if the antipoints of x_2x_3 are a_1b_1, of x_3x_1 are a_2b_2, of x_1x_2 are a_3b_3, $x_1x_2x_3$ being real points, then a_1b_3, a_2b_1, a_3b_2 meet at a circular point I, and a_1b_2, a_2b_3, a_3b_1 meet at J; but on the other hand a_1b_1, a_2b_2, a_3b_3 meet at the circumcenter of $x_1x_2x_3$. It is worth remarking that this theorem is identical with Pascal's theorem for a two-line.*

It is convenient to think of triply perspective triangles as concentric equilateral triangles. That the centers of perspection form a triangle mutual with the two is then obvious; that is, triply perspective 3-points fall into sets of three, or *triads*.

It is further obvious from this metrically canonic case that two triangles A, B fully perspective with a given one T are triply perspective, for if T contain I and J then A, B are equilateral and concentric.

Fig. 1, which is KANTOR's configuration 9_3A, shows a triad of 3-points, and

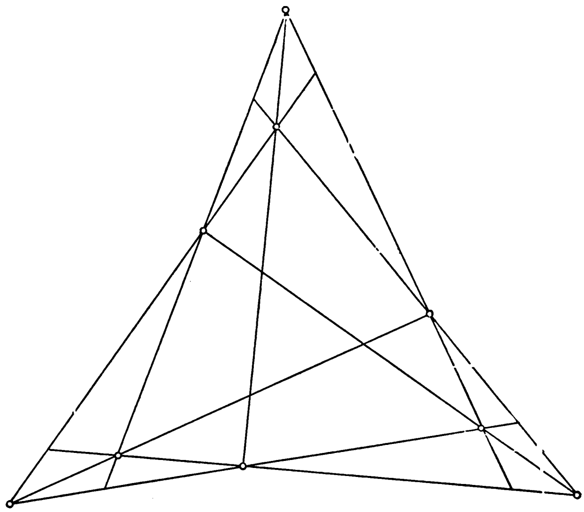

FIG. 1.

also the accompanying triad of 3-lines. Note that there is no such thing as a triad of *triangles* in the present sense.

The coördinates of a 3-point fully perspective with a 3-point T of reference are, if $\omega = \exp.(2\pi i/3)$,

2. 1) $$A: \begin{Vmatrix} a_1 & a_2 & a_3 \\ a_1 & \omega a_2 & \omega^2 a_3 \\ a_1 & \omega^2 a_2 & \omega a_3 \end{Vmatrix}.$$

Three such 3-points ABC form a triad when

2. 2) $$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 = 0,$$ $$a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1 = 0;$$

* Cf. SCHROETER, Mathematische Annalen, vol. 2 (1870), p. 533.

We note from adding these equations that collinear points of ABC are apolar with T.

A triad of triply perspective 3-points obviously belongs to a range of cubic line-curves, whence observing that the equation of A is

2. 3) $$a_1^3\xi_1^3 + a_2^3\xi_2^3 + a_3^3\xi_3^3 - 3a_1a_2a_3\xi_1\xi_2\xi_3 = 0,$$

we infer that the equations

2. 4) $$\left\|\begin{matrix} a_1^3 & a_2^3 & a_3^3 & a_1a_2a_3 \\ b_1^3 & b_2^3 & b_3^3 & b_1b_2b_3 \\ c_1^3 & c_2^3 & c_3^3 & c_1c_2c_3 \end{matrix}\right\| = 0$$

are deducible from (2. 2).

Let A and B coincide. Then the 3-point C is inscribed in the 3-line A. Incidentally it is worth noting how, given A and the inscribed 3-point C, the 3-point T fully perspective with A and C is determined. Let A be the reference-triangle, T be

$$\begin{bmatrix} t_1 & t_2 & t_3 \\ t_1 & \omega t_2 & \omega^2 t_3 \\ t_1 & \omega^2 t_2 & \omega t_3 \end{bmatrix}$$

and C be

$$\begin{bmatrix} 0 & \lambda & 1 \\ 1 & 0 & \mu \\ \nu & 1 & 0 \end{bmatrix}.$$

Then they are perspective in the order given if

$$\lambda t_1 t_3^2 + \omega\mu t_2 t_1^2 + \omega^2 \nu t_3 t_2^2 = \mu\nu t_1 t_2^2 + \omega^2 \nu\lambda t_2 t_3^2 + \omega\lambda\mu t_3 t_1^2,$$

and they are perspective in all ways if

$$\lambda t_1 t_3^2 = \mu t_2 t_1^2 = \nu t_3 t_2^2,$$

or if

$$t_1 = \sqrt[3]{\nu^2\lambda}, \qquad t_2 = \sqrt[3]{\lambda^2\mu}, \qquad t_3 = \sqrt[3]{\mu^2\nu}.$$

Hence *two triangles, one inscribed to the other, may always be projected into equilateral triangles.* It is true that any two triangles may be projected into equilateral triangles, but the investigation does not belong here.

§ 3. *Transference to a cubic surface.*

We consider now the equation

3. 1) $$\Xi_1 x_1^3 + \Xi_2 x_2^3 + \Xi_3 x_3^3 + \Xi_0 x_1 x_2 x_3 = 0,$$

connecting a point x of a plane S_2 and a plane Ξ of a space S_3. Given x, Ξ is on a point X, with coördinates

3. 2) $$X_i = x_i^3, \qquad X_0 = x_1 x_2 x_3 \qquad (i = 1, 2, 3),$$

so that the S_2 maps into the cubic surface Φ:

3. 3) $$X_1 X_2 X_3 = X_0^3 .$$

This surface is in the Brill-Schilling collection (ser. 7, no. 9).

The 3 lines $X_0 = X_i = 0$ whose points are on Φ will be called rays of Φ; the other 3 lines of reference, whose planes are on Φ, will be called axes of Φ.

But (3. 2), if true for x_1, x_2, x_3, is equally true for x_1, ωx_2, $\omega^2 x_3$. Hence a point of Φ represents a 3-point of S_2, fully perspective with the 3-point of reference T.

Given Ξ, x is on a cubic curve ϕ with flexes on the lines of T.

Thus a plane section of Φ maps into a cubic curve, the correspondence being 1 : 3. The collinear points ABC of Φ map into the 9 meets of a pencil of cubics ϕ. Such a pencil of cubics contains (besides the 3-line T) three 3-lines. Hence, collinear points ABC map into triply perspective 3-points. When A and B coincide, C is on S_2 a 3-point on the lines of A. Hence *a tangent plane of* Φ *represents a* 3-*line of* S_2. This may be verified directly; for the 3-line A is

$$\frac{x_1^3}{a_1^3} + \frac{x_2^3}{a_2^3} + \frac{x_3^3}{a_3^3} - \frac{3x_1 x_2 x_3}{a_1 a_2 a_3} = 0,$$

whence

$$\Xi_i = \frac{1}{a_i^3}, \qquad \Xi_0 = -\frac{3}{a_1 a_2 a_3},$$

and

3. 4) $$27\,\Xi_1 \Xi_2 \Xi_3 + \Xi_0^3 = 0,$$

the plane-equation of Φ.

The tangent plane of Φ at A being

$$X_1 a_1^3 + X_2 a_2^3 + X_3 a_3^3 = 3 X_0 a_1 a_2 a_3,$$

we see that to the tangent planes on a point X correspond 3-lines of a line-cubic

3. 5) $$X_1 \xi_1^3 + X_2 \xi_2^3 + X_3 \xi_3^3 = 3 X_0 \xi_1 \xi_2 \xi_3$$

and beginning with this we could reverse the argument of this section.

To the 3 points and 3 planes of Φ on any line corresponds Figure 1.

Since a 3-point and its 3-line are represented by a point of Φ and its tangent plane, to the *triangle* of the plane corresponds the *element* (point and plane thereat) of Φ. The triangle of reference is represented by the point on the axes, and the plane on the rays. The point on the axes will be called T.

To 2 triangles in 4-fold perspective correspond 2 elements whose points are on a plane with an axis of the surface, or (what follows) whose planes are on a point with the opposite ray of the surface.

In particular if a point X of Φ represents the points of one of a system of equilateral triangles with common center, then the middle points of the sides may be represented by the point

$$-\tfrac{1}{2}X_0, \; -\tfrac{1}{8}X_1, \; X_2 \; X_3.$$

§ 4. *The Clebschian of triply perspective 3-points.*

The tetrads of fully perspective triangles (§ 1) such as $TA_1A_2A_3$ are represented by points $TA_1A_2A_3$ on a line, A_i being on Φ. Since the polar of T is a repeated plane, these 4 points are self-apolar.

Thus a plane on T meets the cubic in a cubic curve with the same property; any tangent from T is therefore a stationary tangent, for when of 4 self-apolar points two coincide, three do. Such a cubic curve is sometimes called equianharmonic; as its form is reducible to the sum of 3 cubes, it may be called *catalectic*, after SYLVESTER's use of catalecticant. It is of course one for which the invariant S of degree 4 vanishes; from the present position this is because (1), the polar conic of T is a repeated line cutting out flexes F_i; (2), hence the polar conic of F_i is two lines on T; (3), hence the 4 tangents of the cubic at and from F_i are self-apolar.

If I may call the cubic $(abx)^3$ derived from $(a\xi)^3$ and $(b\xi)^3$ their *Clebschian* —a name of course covering other like cases—then the Clebschian of the 3-points A and B is

$$\begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix} = 0.$$

This is a catalectic cubic apolar to A and B; *it is represented simply by the plane TAB.*

It is equally the Clebschian of the cubics of the web built on T, A, B; to express its equation symmetrically in terms of the triad ABC we have from (2. 4)

$$a_2^3b_3^3 - a_3^3b_2^3 = \rho(a_1^3\cdot b_1b_2b_3 - b_1^3\cdot a_1a_2a_3)$$

$$= \rho a_1b_1(a_1^2b_2b_3 - b_1^2a_2a_3)$$

or from (2. 2)

$$= \rho' a_1b_1c_1.$$

Hence the Clebschian is

$$a_1b_1c_1x_1^3 + a_2b_2c_2x_2^3 + a_3b_3c_3x_3^3 = 0.$$

It is worth noting that if on a catalectic cubic ϕ we take collinear points ABC and join to one of the Cayleyan points, we get 6 other points on ϕ which lie on two lines. For if we cut

$$x_1^3 + x_2^3 + x_3^3 = 0$$

by $(\beta x) = 0$ and join to $(0\,0\,1)$ we get

$$\beta_3^3(x_1^3 + x_2^3) = (\beta_1 x_1 + \beta_2 x_2)^3,$$

and the points where these lines meet ϕ lie on the 3 lines

$$(\beta_3 x_3)^3 + (\beta_1 x_1 + \beta_2 x_2)^3 = 0,$$

one of course being β itself. These 3 lines meet on the Hessian line $x_3 = 0$. Thus if

$$TA_1A_2A_3, \qquad TB_1B_2B_3, \qquad TC_1C_2C_3,$$

are tetrads of fully perspective 3-points, of which $A_1B_1C_1$ are a triad then (when the ordering is right) $A_2B_2C_2$ and $A_3B_3C_3$ are also triads and the Clebschian of any two of the 3-points $A_iB_jC_k$ is the Clebschian of every two.

§ 5. *The Clebschian of any two 3-points.*

Hitherto only 3-points fully perspective with a given one have been considered. With regard to two unrestricted 3-points I wish to prove here one proposition; I hope to discuss the general question in another paper. The proposition arises from the Clebschian.

The Clebschian of any two cubics,

5. 1, 1a) $$X_1\xi_1^3 + X_2\xi_2^3 + X_3\xi_3^3 + 6X_0\xi_1\xi_2\xi_3 = 0, \; (a\xi)^3 = 0,$$

contains no term in $x_1x_2x_3$. Hence calling the reference 3-point *syzygetic* with (5. 1), the four 3-points syzygetic with either cubic are apolar to the Clebschian; as of course are also the cubics themselves. In particular let $X_1 = X_2 = X_3 = 0$; then the Clebschian of

$$\xi_1\xi_2\xi_3 = 0, \qquad (a\xi)^3 = 0,$$

is

5. 2) $$(a_2x_3 - a_3x_2)(a_3x_1 - a_1x_3)(a_1x_2 - a_2x_1) = 0,$$

and the 3-point $\xi_1\xi_2\xi_3$ is both on and apolar to the Clebschian. Thus the Clebschian of two 3-points is subject to the 8 conditions of having two such 3-points given, and to a further condition.

If the 8 conditions are independent, there is a pencil of such cubics, and they meet in a third 3-point. This is to be investigated in §§ 6 and 7. The 8 conditions are not independent if the two 3-points are triply perspective, for then, as is easily verified, a cubic on both and apolar to one is apolar to the other.

And if the two 3-points are fully perspective, any cubic on both is apolar to both; and *the Clebschian is the whole plane.*

A cubic ϕ being given by

$$x = p(u), \qquad y = p'(u),$$

the condition that a 3-point u_i be both on and apolar is seen by reference to HALPHEN (*Fonctions Elliptiques*, vol. 2, p. 428) to be

$$\text{5. 3)} \qquad \zeta(u_2 - u_3) + \zeta(u_3 - u_1) + \zeta(u_1 - u_2) = 0,$$

whence *the projection of such a 3-point from a point of ϕ on to ϕ is again such a 3-point*, and the sides of such a 3-point cut out another one. By independent proof of the former result the last formula is seen at once. For let u_1 be the flex 0, then u_2 and u_3 are apolar with the polar conic of the flex, i. e., $p'(u_2) = p'(-u_3)$ and this is another form of (5. 3) when $u_1 = 0$.

On the Clebschian of two given 3-points u_i and v_i a large number of points can be at once constructed.

Thus if we complete the 3-point $v_1 ab$ fully perspective with u_i, and cross-join ab and $v_2 v_3$ we obtain two points of the curve — 12 points in all.

And if we take a point e, such that ev_2, ev_3 are the Hessian of the lines eu_i* then e is a point of ϕ.

And if the conic $u_1 u_2 u_3 v_2 v_3$ meets ϕ again at c, and $u_1 u_2 u_3 v_2 v_3 d$ are apolar along the conic, then cdv_1 are on a line.

And the points where $u_2 u_3$ and $u_i v_j$ meet ϕ may be readily determined.

§ 6. *Three-lines on and apolar to a deltoid.*

Taking now the pencil of cubics of § 5, we select a rational curve of the pencil. For convenience of statement† I take this rational cubic as a deltoid, Δ^3,

$$t^3 - xt^2 + yt - 1 = 0.$$

The question is first: When is a 3-line on this deltoid apolar with it? The line-equation of the curve, given by $\xi_1 = t^3 - 1$, $\xi_2 = -t^2$, $\xi_3 = t$, is

$$\xi_2^3 + \xi_3^3 = \xi_1 \xi_2 \xi_3.$$

Hence lines $\xi\eta\zeta$ are apolar with it when

$$6(\xi_2\eta_2\zeta_2 + \xi_3\eta_3\zeta_3) = \Sigma\xi_1\eta_2\zeta_3,$$

*See Bulletin of the American Mathematical Society, vol. 1 (1895), p. 124.

† For a sketch of the vector treatment I refer to my paper on *Orthocentric properties of the plane n-line*, Transactions, vol. 4 (1903), pp. 1–12.

and if the parameters of these lines are given by

$$t_i^3 - s_1 t_i^2 + s_2 t_i - s_3 = 0,$$

then

$$6(s_3^2 - s_3) = \sum t_1^2 t_2 (t_3^3 - 1) = (s_3 - 1) \sum t_1^2 t_2;$$

that is, discarding the case of lines on a point for which $s_3 = 1$,

$$6 s_3 = \sum t_1^2 t_2,$$

or

6. 1) $$s_1 s_2 = 9 s_3.$$

Now s_1 is the circumcenter; hence *the circumcenter of the 3-line is on the cusp-circle.*

There are then in the deltoidal case no real non-degenerate 3-lines both on

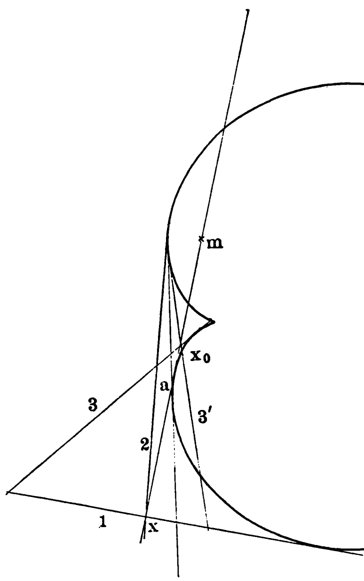

FIG. 2.

and apolar. There are such in the cardioidal case; to construct them we write the equation for t_3:

$$t^2(t_1 + t_2) + t_1 t_2 (t_1 + t_2) + t(t_1^2 + t_2^2 - 6 t_1 t_2) = 0,$$

where

$$tt' = t_1 t_2 = 1/t_0,$$

say. Then

$$(t + t' + t_1 + t_2)(t_1 + t_2) = s/t_0,$$

so that if x is the known point $t_0 + t_1 + t_2$ where 2 of the 3-lines meet, x_0 is the point $t_0 + t + t'$ on both of the solution-lines, and $x + x_0 = 2m$, then

6. 2) $$(m - t_0)(x - t_0) = 4/t_0,$$

an involution with double points where the line t_0 meets the curve again.

Fig. 2 shows two such 3-lines, namely, 123 and 123′.

§ 7. *The triplicity of 3-points.*

Let a_i and b_i be two such 3-lines of the deltoid. Their Clebschian is

$$\sum_{(b)} |\xi a_1 b_1| \, |\xi a_2 b_2| \, |\xi a_3 b_3| = 0$$

summed for cyclical permutations of b_i. The common lines of this and Δ^3 are given by

$$\sum_{(b)} |t^3 - 1, a_1^2, b_1| \, |t^3 - 1, a_2^2, b_2| \, |t^3 - 1, a_3^2, b_3| = 0.$$

But

$$\begin{vmatrix} t^3 - 1 & t^2 & t \\ a^3 - 1 & a^2 & a \\ b^3 - 1 & b^2 & b \end{vmatrix} = (tab - 1)(t - a)(t - b)(a - b).$$

Hence the third 3-line is given by

7. 1) $$\sum_{(b)} (ta_1 b_1 - 1)(ta_2 b_2 - 1)(ta_3 b_3 - 1)(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 0,$$

or if

$$I = \sum (a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 3(q_3 - r_3) - q_2 r_1 + q_1 r_2,$$

where q_i, r are the coefficients of a_i and b_i, e. g., $q_1 = a_1 + a_2 + a_3$, then

$$(t^3 q_3 r_3 - 1) I - t^2 \{ \ \} + t \sum (a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 0.$$

Hence

$$q_3 r_3 s_3 = 1,$$

$$I s_2 / s_3 = (q_3 - r_3) q_1 r_1 - 3 q_3 r_1^2 + 3 r_3 q_1^2 + 9(q_3 r_2 - r_3 q_2) - q_2 r_2 (q_1 - r_1),$$

or, since $9 q_3 = q_1 q_2$, $9 r_3 = r_1 r_2$,

$$I s_2 / s_3 = q_1 r_1 [q_3 - r_3 - \tfrac{1}{3}(q_2 r_1 - q_1 r_2)] = \tfrac{1}{3} I q_1 r_1.$$

Thus either $I = 0$, that is the 3-lines are apolar to each other along the curve, in which case the algebra shows that the Clebschian is Δ^3, or *in general*

$$s_2 / s_3 = q_1 r_1 / 3,$$

whence

7. 2) $$q_1 r_1 s_1 = 27.$$

That is, the circumcenters, which we know are on the cusp-circle, are along that circle apolar with the cusps.

Hence the relation of the three 3-lines is mutual, and returning to the dualistic statement:

Two general 3-points of a plane determine uniquely and mutually a third; or the 3-points of a plane projectively considered fall into sets of three — say form a *triplicity*. An exceptional case is that of two fully perspective 3-points — these are *neutral*, i. e., determine no 3-point. Another apparently exceptional case is that of two triply perspective 3-points, but it is likely that by knowing further properties of the triplicity we should find that the triads of § 2 do belong to it.

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